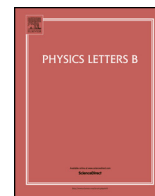



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Energy and transverse momentum fluctuations in the equilibrium quantum systems


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ABSTRACT

The fluctuations in the ideal quantum gases are studied using the strongly intensive measures $\Delta[A, B]$ and $\Sigma[A, B]$ defined in terms of two extensive quantities A and B . In the present Letter, these extensive quantities are taken as the motional variable, $A = X$, the system energy E or transverse momentum P_T , and number of particles, $B = N$. This choice is most often considered in studying the event-by-event fluctuations and correlations in high energy nucleus–nucleus collisions. The recently proposed special normalization ensures that Δ and Σ are dimensionless and equal to unity for fluctuations given by the independent particle model. In statistical mechanics, the grand canonical ensemble formulation within the Boltzmann approximation gives an example of independent particle model. Our results demonstrate the effects due to the Bose and Fermi statistics. Estimates of the effects of quantum statistics in the hadron gas at temperatures and chemical potentials typical for thermal models of hadron production in high energy collisions are presented. In the case of massless particles and zero chemical potential the Δ and Σ measures are calculated analytically.

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1. Introduction

Experimental and theoretical studies of the event-by-event (e-by-e) fluctuations in nucleus–nucleus ($A + A$) collisions give new information about properties of the strongly interacting matter and its phases. A possibility to observe signatures of the QCD matter critical point inspired the energy and system size scan program of the NA61/SHINE Collaboration at the SPS CERN [1] and the low energy scan program of the STAR and PHENIX Collaborations at the RHIC BNL [2]. In these studies one measures and then compares the e-by-e fluctuations in collisions of different nuclei at different collision energies. The average sizes of the created physical systems and their e-by-e fluctuations are expected to be rather different [3]. This strongly affects the observed hadron fluctuations, i.e. the measured quantities would not describe the local physical properties of the system but rather reflect the system size fluctuations. For instance, $A + A$ collisions with different centralities may produce a system with approximately the same local properties (e.g., the same temperature and baryonic chemical potential) but with the volume changing significantly from interaction to interaction. Note that in high energy collisions the average volume of created matter and its variations from collision to collision are

usually out of experimental control (i.e. these volume variations are difficult or even impossible to measure).

In the statistical mechanics the extensive quantity A is proportional to the system volume V , whereas intensive quantity has a fixed finite value in the thermodynamical limit $V \rightarrow \infty$. The intensive quantities are used to describe the local properties of a physical system. In particular, an equation of state of the matter is usually formulated in terms of the intensive physical quantities, e.g., the pressure is considered as a function of temperature and chemical potentials. In the statistical systems outside of the phase transition regions, a mean value of fluctuating extensive quantity, $\langle A \rangle$, and its variance, $\text{Var}(A) = \langle A^2 \rangle - \langle A \rangle^2$, are both proportional to the volume V in the limit of large volumes. The scaled variance,

$$\omega[A] = \frac{\langle A^2 \rangle - \langle A \rangle^2}{\langle A \rangle}, \quad (1)$$

is therefore an intensive quantity. However, the scaled variance being an intensive quantity depends on the system size fluctuations.

Strongly intensive quantities introduced in Ref. [4] are independent of the average volume and of volume fluctuations. These quantities were suggested for and are used in studies of e-by-e fluctuations of hadron production in $A + A$ collisions. Strongly intensive measures of fluctuations are defined in terms of two arbitrary extensive quantities A and B . In the present study we

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consider a pair of extensive variables – the motional extensive variable $X = x_1 + \dots + x_N$ as a sum of single particle variables x_j , with $j = 1, \dots, N$, and number of particles N . These measures were recently studied within the UrQMD simulations in Ref. [5]. The case of two hadron multiplicities A and B in $A + A$ collisions has been considered within the HSD transport model in Ref. [6]. At the beginning we identify a single particle variable x with the particle energy ϵ and then consider the particle transverse momentum p_T .

The strongly intensive measure $\Delta[X, N]$ and $\Sigma[X, N]$ are defined as [4]:

$$\Delta[X, N] = \frac{1}{C_\Delta} [\langle N \rangle \omega[X] - \langle X \rangle \omega[N]], \quad (2)$$

$$\Sigma[X, N] = \frac{1}{C_\Sigma} [\langle N \rangle \omega[X] + \langle X \rangle \omega[N] - 2(\langle XN \rangle - \langle X \rangle \langle N \rangle)], \quad (3)$$

where C_Δ and C_Σ are the normalization factors, and the scaled variances $\omega[X]$ and $\omega[N]$ are given by Eq. (1).

In Ref. [7] a special normalization for the strongly intensive measures Δ and Σ has been proposed. Namely, the properly normalized strongly intensive quantities assume the value *one* for fluctuations given by the independent particle model (IPM). For the X and N extensive quantities the proposed normalization reads [7]:

$$C_\Delta = C_\Sigma = \omega[x] \cdot \langle N \rangle, \quad \omega[x] \equiv \frac{\overline{x^2} - \bar{x}^2}{\bar{x}}. \quad (4)$$

Note that the overline denotes averaging over a single particle inclusive distribution, whereas $\langle \dots \rangle$ represents averaging over multiparticle states of the system.

The first strongly intensive measure for fluctuations, the so-called Φ measure, was introduced a long time ago in Ref. [8]. The Φ quantity for the ideal quantum gases was considered in Ref. [9]. There were numerous attempts to use the Φ measure describing fluctuations in experimental data [10] and models [11]. In general, however, Φ is a dimensional quantity and it does not have a characteristic scale for a quantitative analysis of e-by-e fluctuations for different observables. Note that the latter properties were clearly disturbing. The Φ measure can be expressed in terms of Σ [4]. A presence of additional fluctuation measure Δ and utilization of special normalization conditions for both Δ and Σ give essential advantages in application to the data analysis in $A + A$ collisions.

In the present Letter we study the strongly intensive measures (2) and (3) with normalization factors (4) for the relativistic ideal quantum gases in the grand canonical ensemble. The Letter is organized as follows. In Section 2 we calculate the $\Delta[X, N]$ and $\Sigma[X, N]$ quantities for the ideal quantum gases in the grand canonical ensemble. Analytical and numerical results suitable for the hadron gas created in $A + A$ collisions are presented in Section 3. A summary in Section 4 closes the article.

2. Ideal quantum gas

The grand canonical ensemble (GCE) partition function reads:

$$\mathcal{E}(V, T, \lambda) = \sum_N \sum_\alpha \lambda^N \exp(-\beta E_\alpha), \quad (5)$$

where V is the system volume, $\beta \equiv T^{-1}$ is the inverse system temperature, $\lambda \equiv \exp(\beta\mu)$ denotes the fugacity and μ the chemical potential. The index α numerates the system quantum states, and N is the number of particles. The ensemble average values of the k th moments ($k = 1, 2, \dots$) of any state quantity A are calculated as:

$$\langle A^k \rangle = \frac{1}{\mathcal{E}} \sum_N \sum_\alpha A^k \lambda^N \exp(-\beta E_\alpha). \quad (6)$$

The GCE partition function (5) can be presented in the form

$$\mathcal{E} = \exp \left\{ V \eta^{-1} d \int \frac{d^3 p}{(2\pi)^3} \ln[1 + \eta \lambda \exp(-\beta \epsilon)] \right\}, \quad (7)$$

where d is the number of particle internal degrees of freedom and $\epsilon \equiv \sqrt{m^2 + \mathbf{p}^2}$ is the particle energy with m being the particle mass and \mathbf{p} its momentum. The values $\eta = -1$ and $\eta = 1$ correspond to the Bose and Fermi statistics, respectively, whereas $\eta = 0$ to the Boltzmann approximation. Using the presentation (7) one can calculate the averages (6) for the 1st and 2nd moments of the energy E and number of particles N :

$$\begin{aligned} \langle N \rangle &= \frac{1}{\mathcal{E}} \lambda \frac{\partial \mathcal{E}}{\partial \lambda} = V \rho, \\ \rho &\equiv d \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\lambda^{-1} \exp(\epsilon/T) + \eta}, \end{aligned} \quad (8)$$

$$\begin{aligned} \langle N^2 \rangle &= \frac{1}{\mathcal{E}} \left(\lambda \frac{\partial}{\partial \lambda} \right)^2 \mathcal{E} = V^2 \rho^2 + V I_N, \\ I_N &\equiv d \int \frac{d^3 p}{(2\pi)^3} \frac{\lambda^{-1} \exp(\epsilon/T)}{[\lambda^{-1} \exp(\epsilon/T) + \eta]^2}, \end{aligned} \quad (9)$$

$$\begin{aligned} \langle E \rangle &= -\frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial \beta} = V \varepsilon, \\ \varepsilon &\equiv d \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon}{\lambda^{-1} \exp(\epsilon/T) + \eta}, \end{aligned} \quad (10)$$

$$\begin{aligned} \langle E^2 \rangle &= \frac{1}{\mathcal{E}} \frac{\partial^2}{\partial \beta^2} \mathcal{E} = \langle E \rangle^2 + V I_E, \\ I_E &\equiv d \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon^2 \lambda^{-1} \exp(\epsilon/T)}{[\lambda^{-1} \exp(\epsilon/T) + \eta]^2}, \end{aligned} \quad (11)$$

$$\begin{aligned} \langle EN \rangle &= -\frac{1}{\mathcal{E}} \frac{\partial}{\partial \beta} \lambda \frac{\partial}{\partial \lambda} \mathcal{E} = \langle N \rangle \langle E \rangle + V I_{EN}, \\ I_{EN} &\equiv d \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon \lambda^{-1} \exp(\epsilon/T)}{[\lambda^{-1} \exp(\epsilon/T) + \eta]^2}, \end{aligned} \quad (12)$$

where $\rho \equiv \langle N \rangle / V$ and $\varepsilon \equiv \langle E \rangle / V$ denote the particle number density and the energy density, respectively.

From Eqs. (8)–(12) one finds for the scaled variances:

$$\omega[N] \equiv \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = \frac{I_N}{\rho}, \quad \omega[E] \equiv \frac{\langle E^2 \rangle - \langle E \rangle^2}{\langle E \rangle} = \frac{I_E}{\varepsilon}. \quad (13)$$

They describe the fluctuations of the number of particles and the system energy at fixed volume V . The scaled variances in Eq. (13) are intensive quantities, they depend only on T and μ . The quantities (13) are independent of the particle degeneracy factor d . Note that there is a (positive) correlation between the energy E and particle number N :

$$\langle EN \rangle - \langle E \rangle \langle N \rangle = V I_{EN} > 0. \quad (14)$$

The moments of single particle energy ϵ ($k = 1, 2$) are

$$\bar{\epsilon}^k = \frac{d}{\rho} \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon^k}{\lambda^{-1} \exp(\epsilon/T) + \eta}, \quad (15)$$

and the scaled variance $\omega[\epsilon]$ is

$$\omega[\epsilon] = \frac{\bar{\epsilon}^2 - \bar{\epsilon}^2}{\bar{\epsilon}}. \quad (16)$$

Calculating $\Delta[E, N]$ and $\Sigma[E, N]$ according to Eqs. (2)–(4) one obtains:

$$\begin{aligned}\Delta[E, N] &= \frac{1}{\omega[\epsilon]} [\omega[E] - \bar{\epsilon} \cdot \omega[N]] \\ &= \frac{1}{\omega[\epsilon] \cdot \rho} \left[\frac{I_E}{\bar{\epsilon}} - \bar{\epsilon} \cdot I_N \right],\end{aligned}\quad (17)$$

$$\begin{aligned}\Sigma[E, N] &= \frac{1}{\omega[\epsilon]} \left[\omega[E] + \bar{\epsilon} \cdot \omega[N] - 2 \frac{I_{EN}}{\rho} \right] \\ &= \frac{1}{\omega[\epsilon] \cdot \rho} \left[\frac{I_E}{\bar{\epsilon}} + \bar{\epsilon} \cdot I_N - 2 I_{EN} \right].\end{aligned}\quad (18)$$

Note that our choice of the normalization (4) makes $\Delta[E, N]$ and $\Sigma[E, N]$ dimensionless. These quantities are also independent of the degeneracy factor d .

The GCE within Boltzmann approximation satisfies the assumptions of IPM. Thus, one expects for the Boltzmann gas

$$\Delta^{\text{Boltz}}[E, N] = \Sigma^{\text{Boltz}}[E, N] = 1. \quad (19)$$

This can be easily proven, as for the $\eta = 0$ one finds from Eqs. (8)–(12):

$$\begin{aligned}\omega[N] &= \frac{I_N}{\rho} = 1, \quad \omega[E] = \frac{I_E}{\bar{\epsilon} \cdot \rho} = \frac{\bar{\epsilon}^2}{\bar{\epsilon}}, \\ \frac{\langle EN \rangle - \langle E \rangle \langle N \rangle}{\langle N \rangle} &= \frac{I_{EN}}{\rho} = \bar{\epsilon},\end{aligned}\quad (20)$$

and Eqs. (17) and (18) are transformed to Eq. (19). Using (8)–(12) the following general relations can be proven

$$\Delta^{\text{Bose}}[E, N] < \Delta^{\text{Boltz}} = 1 < \Delta^{\text{Fermi}}[E, N], \quad (21)$$

$$\Sigma^{\text{Fermi}}[E, N] < \Sigma^{\text{Boltz}} = 1 < \Sigma^{\text{Bose}}[E, N], \quad (22)$$

i.e. Bose statistics makes $\Delta[E, N]$ to be smaller and $\Sigma[E, N]$ larger than unity, whereas Fermi statistics works in exactly opposite way.

The strongly intensive measures Δ and Σ are independent of the volume and of its fluctuations. This is valid within the GCE, when the temperature and chemical potentials are volume independent. In a presence of volume fluctuations, the second moments $\langle E^2 \rangle$, $\langle N^2 \rangle$, and $\langle EN \rangle$ will include the terms proportional to $\langle V^2 \rangle$ which describe the contributions of the volume fluctuations to the fluctuations of E and N . The full averaging will then include both the GCE averaging (8)–(12) at fixed volume V and an additional averaging over the volume fluctuations. The scaled variances (13) do not depend on the average volume of the system, i.e. they are intensive quantities. However, they do depend on the volume fluctuations, i.e. the scaled variances are not strongly intensive quantities. On the other hand, the straightforward calculations demonstrate [4] that contributions from the volume fluctuations to Δ and Σ are canceled out and Eqs. (17), (18) remain valid, i.e. Δ and Σ are indeed the strongly intensive measures in the GCE.

3. Hadron gas

In this section we consider the measures Δ and Σ for the hadron gas with thermodynamical parameters typical for the thermal models of A + A collisions.

3.1. Massless particles

We start from the system with $m = \mu = 0$. In the case, the calculations of quantities entering (8)–(12) can be performed analytically:

$$\begin{aligned}\rho &= \frac{d}{2\pi^2} \int_0^\infty p^2 dp \frac{1}{\exp(p/T) \pm 1} = \frac{d\zeta(3)}{\pi^2} \left(\frac{3/4}{1} \right) T^3 \\ &\cong d \left(\frac{0.091}{0.122} \right) T^3,\end{aligned}\quad (23)$$

$$\begin{aligned}\bar{\epsilon} &= \frac{d}{2\pi^2 \rho} \int_0^\infty p^2 dp \frac{p}{\exp(p/T) \pm 1} = \frac{3\zeta(4)}{\zeta(3)} \left(\frac{7/6}{1} \right) T \\ &\cong \left(\frac{3.152}{2.701} \right) T,\end{aligned}\quad (24)$$

$$\begin{aligned}\bar{\epsilon}^2 &= \frac{d}{2\pi^2 \rho} \int_0^\infty p^2 dp \frac{p^2}{\exp(p/T) \pm 1} = \frac{12\zeta(5)}{\zeta(3)} \left(\frac{5/4}{1} \right) T^2 \\ &\cong \left(\frac{12.941}{10.352} \right) T^2,\end{aligned}\quad (25)$$

$$\begin{aligned}I_N &= \frac{d}{2\pi^2} \int_0^\infty p^2 dp \frac{\exp(p/T)}{[\exp(p/T) \pm 1]^2} = \frac{d\zeta(2)}{\pi^2} \left(\frac{1/2}{1} \right) T^3 \\ &\cong d \left(\frac{0.083}{0.167} \right) T^3,\end{aligned}\quad (26)$$

$$\begin{aligned}I_{EN} &= \frac{d}{2\pi^2} \int_0^\infty p^2 dp \frac{p \exp(p/T)}{[\exp(p/T) \pm 1]^2} = \frac{3d\zeta(3)}{\pi^2} \left(\frac{3/4}{1} \right) T^4 \\ &\cong d \left(\frac{0.274}{0.365} \right) T^4,\end{aligned}\quad (27)$$

$$\begin{aligned}I_E &= \frac{d}{2\pi^2} \int_0^\infty p^2 dp \frac{p^2 \exp(p/T)}{[\exp(p/T) \pm 1]^2} = \frac{12d\zeta(4)}{\pi^2} \left(\frac{7/8}{1} \right) T^5 \\ &\cong d \left(\frac{1.151}{1.316} \right) T^5,\end{aligned}\quad (28)$$

where $\zeta(s)$ is the Riemann zeta function: $\zeta(2) = \pi^2/6 \cong 1.645$, $\zeta(3) \cong 1.202$, $\zeta(4) = \pi^4/90 \cong 1.082$, $\zeta(5) \cong 1.037$. The upper case in Eqs. (23)–(28) and in equations below corresponds to fermions ($\eta = +1$) and the lower one to bosons ($\eta = -1$). From these equations it follows:

$$\begin{aligned}\omega[\epsilon] &\cong \left(\frac{0.954}{1.132} \right) T, \quad \bar{\epsilon} \cdot \frac{I_N}{\rho} \cong \left(\frac{2.875}{3.697} \right) T, \\ \frac{I_E}{\rho \bar{\epsilon}} &= 4T, \quad \frac{I_{EN}}{\rho} = 3T.\end{aligned}\quad (29)$$

Finally, one obtains

$$\Delta[E, N] \cong \left(\frac{1.179}{0.268} \right), \quad \Sigma[E, N] \cong \left(\frac{0.917}{1.499} \right). \quad (30)$$

The strongly intensive measures $\Delta[E, N]$ and $\Sigma[E, N]$ (30) possess the values which are independent of T . This is evident as the temperature is the only dimensional variable for the system with $m = \mu = 0$, and Δ and Σ measures are dimensionless quantities due to our normalization.

In the Boltzmann approximation $\eta = 0$, the integrals in Eqs. (23)–(28) are reduced to $\int_0^\infty dx x^k \exp(-x) = k!$ with $k = 2, 3, 4$. One obtains

$$\omega[\epsilon] = T, \quad \bar{\epsilon} \cdot \frac{I_N}{\rho} = 3T, \quad \frac{I_E}{\rho \bar{\epsilon}} = 4T, \quad \frac{I_{EN}}{\rho} = 3T, \quad (31)$$

and Eq. (19) is satisfied.

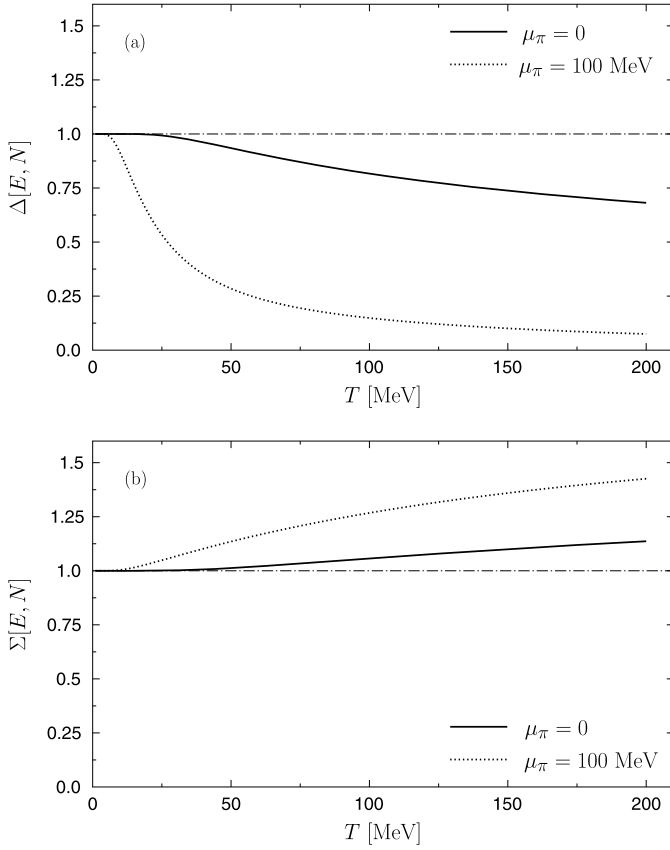


Fig. 1. The (a) $\Delta[E, N]$ and (b) $\Sigma[E, N]$ for the pion gas as the functions of T . The solid lines correspond to $\mu_\pi = 0$ and dotted lines to $\mu_\pi = 100$ MeV. The horizontal dashed lines show the Boltzmann approximation (19) equal to 1.

3.2. Pion gas

The pion gas corresponds to the Bose statistics ($\eta = -1$) and $m_\pi \cong 140$ MeV. We consider the chemical equilibrium pion gas ($\mu_\pi = 0$) and one example of chemical non-equilibrium ($\mu_\pi = 100$ MeV). Calculating the quantities ρ , $\bar{\epsilon}$, $\bar{\epsilon}^2$, I_E , I_N , I_{EN} according to Eqs. (8)–(12) and $\omega[\epsilon]$ with Eq. (16) one obtains $\Delta[E, N]$ by Eq. (17) and $\Sigma[E, N]$ by Eq. (18). The dependence $\Delta[E, N]$ and $\Sigma[E, N]$ on the temperature is shown in Fig. 1. The two lines are presented: the solid line for $\mu_\pi = 0$ and the dotted line for $\mu_\pi = 100$ MeV. The horizontal line in Fig. 1(a) and (b) corresponds to the Boltzmann approximation (19). This approximation appears to be always valid at $T \ll m_\pi$. In the ultra-relativistic limit $T \gg m_\pi$ the results from Eq. (30) are approached, i.e. approximately 0.27 for $\Delta[E, N]$ and 1.5 for $\Sigma[E, N]$. Note that solid and dotted lines for $\Delta[E, N]$ verge towards their infinite temperature limit 0.27 from above and from below, respectively. Therefore, $\Delta[E, N]$ at $\mu_\pi = 0$ has a minimum for an intermediate T value. However, the ultra-relativistic limit for pions has of course only a mathematical meaning, the hadron gas does not exist at $T > 200$ MeV.

The typical freeze-out temperatures in statistical and hydrodynamical models of A + A collisions are $T = 130$ –170 MeV. In this temperature region, the deviations of $\Delta[E, N]$ and $\Sigma[E, N]$ from the IPM results (19) are quite significant, about 25% and 10%, respectively. These deviations are strongly enlarged for the chemical non-equilibrium pion gas with $\mu_\pi > 0$. The Bose statistics lead to the singular behavior of fluctuations at $\mu_\pi \rightarrow m_\pi$, which corresponds to the Bose–Einstein condensation of pions. We do not touch this problem in the present study. For the fluctuations of pion multiplicity this was considered in Ref. [12].

In applications to A + A collisions one should however take into account that a substantial fraction of the final state pions come from the resonance decays. These pions do not ‘feel’ [9] the Bose statistics, and thus the Bose statistics contribution to $\Delta[E, N]$ and $\Sigma[E, N]$ is reduced.

The strongly intensive measures $\Delta[E, N]$ (17) and $\Sigma[E, N]$ (18) include the contributions from the energy fluctuations $\omega[E] = I_E/(\rho\bar{\epsilon})$, particle number fluctuations $\bar{\epsilon} \cdot \omega[N] = \bar{\epsilon} \cdot I_N/\rho$, and correlations I_{EN}/ρ between E and N . The average particle energy can be calculated as $\bar{\epsilon} = \langle E \rangle / \langle N \rangle$. The suggested normalization requires also a knowledge of $\bar{\epsilon}^2$ to find the inclusive energy fluctuations $\omega[\epsilon]$ (16). For any physical quantity A in the pion gas we introduce the ratio

$$R(A) = \frac{A^{\text{Bose}}}{A^{\text{Boltz}}}, \quad (32)$$

where A^{Bose} and A^{Boltz} are calculated for Bose statistics and within the Boltzmann approximation, respectively. The Bose and Boltzmann A-values will be calculated at the same T and μ_π values. In Fig. 2 we show the ratios R of the pion gas quantities $\omega[\epsilon]$, $\omega[E]$, $\bar{\epsilon} \cdot \omega[N]$, and I_{EN}/ρ to their Boltzmann approximations. The later can be found by taking $\eta = 0$ in Eqs. (8)–(12). Deviations of the ratios in Fig. 2 from unity are due to the Bose statistics effects for the corresponding physical quantity. One observes that both $\omega[E]$ and I_{EN}/ρ at $\mu_\pi = 0$ are approximately insensitive to the Bose effects. The Bose effects for $\Delta[E, N]$ and $\Sigma[E, N]$ seen in Fig. 1 are mostly due to the quantum statistics contribution to $\bar{\epsilon} \cdot \omega[N]$ and $\omega[\epsilon]$. Particularly, rather large values of $R(\bar{\epsilon} \cdot \omega[N])$ at $\mu_\pi = 100$ MeV and large T are seen in Fig. 2(c). Just these large values are responsible for a suppression of $\Delta[E, N]$ and enhancement of $\Sigma[E, N]$ shown by dotted lines in Fig. 1(a) and (b), respectively. The above observation is also supported by the analytical calculations at $m = \mu = 0$ and becomes even stronger. A comparison of Eq. (29) and Eq. (31) demonstrate that for $m = \mu = 0$ the values of $\omega[\epsilon]$ and $\bar{\epsilon} I_N/\rho$ are sensitive to the effects of quantum statistics whereas $I_E/(\rho\bar{\epsilon})$ and I_{EN}/ρ are not.

The Bose statistics for the pion gas is the main source of quantum statistics effects in the hadron gas with parameters T and μ typical for the hadron system created in A + A collisions. The proton gas corresponds to Fermi statistics ($\eta = 1$) and $m = m_p \cong 938$ MeV. The proton chemical potential is approximately equal to the baryon chemical potential μ_B (additional contribution due to electric chemical potential is negligible in high energy collisions). The effects of quantum statistics in the proton gas increase with increasing of both T and μ_B . However, the T and μ_B values in the hadron gas are correlated: at small energies of A + A collisions, large μ_B and small T values appear, whereas with increasing of collision energy T moves to its maximum of about 170 MeV and simultaneously μ_B approaches to zero. For typical T and μ_B values we find for the proton gas

$$\Delta[E, N] \cong 1.030, \quad \Sigma[E, N] \cong 0.997, \quad \text{at } T = 150 \text{ MeV}, \mu_B = 300 \text{ MeV}, \quad (33)$$

$$\Delta[E, N] \cong 1.040, \quad \Sigma[E, N] \cong 0.997, \quad \text{at } T = 100 \text{ MeV}, \mu_B = 500 \text{ MeV}. \quad (34)$$

A deviation of the Δ and Σ quantities from 1 in an ideal gas within GCE is due to the quantum statistics. From Eqs. (33), (34) one concludes that Fermi statistics effects for the proton gas are quite small for typical T and μ_B values in the hadron gas: they give only a few percent contribution to $\Delta[E, N]$ and almost negligible contribution to $\Sigma[E, N]$.

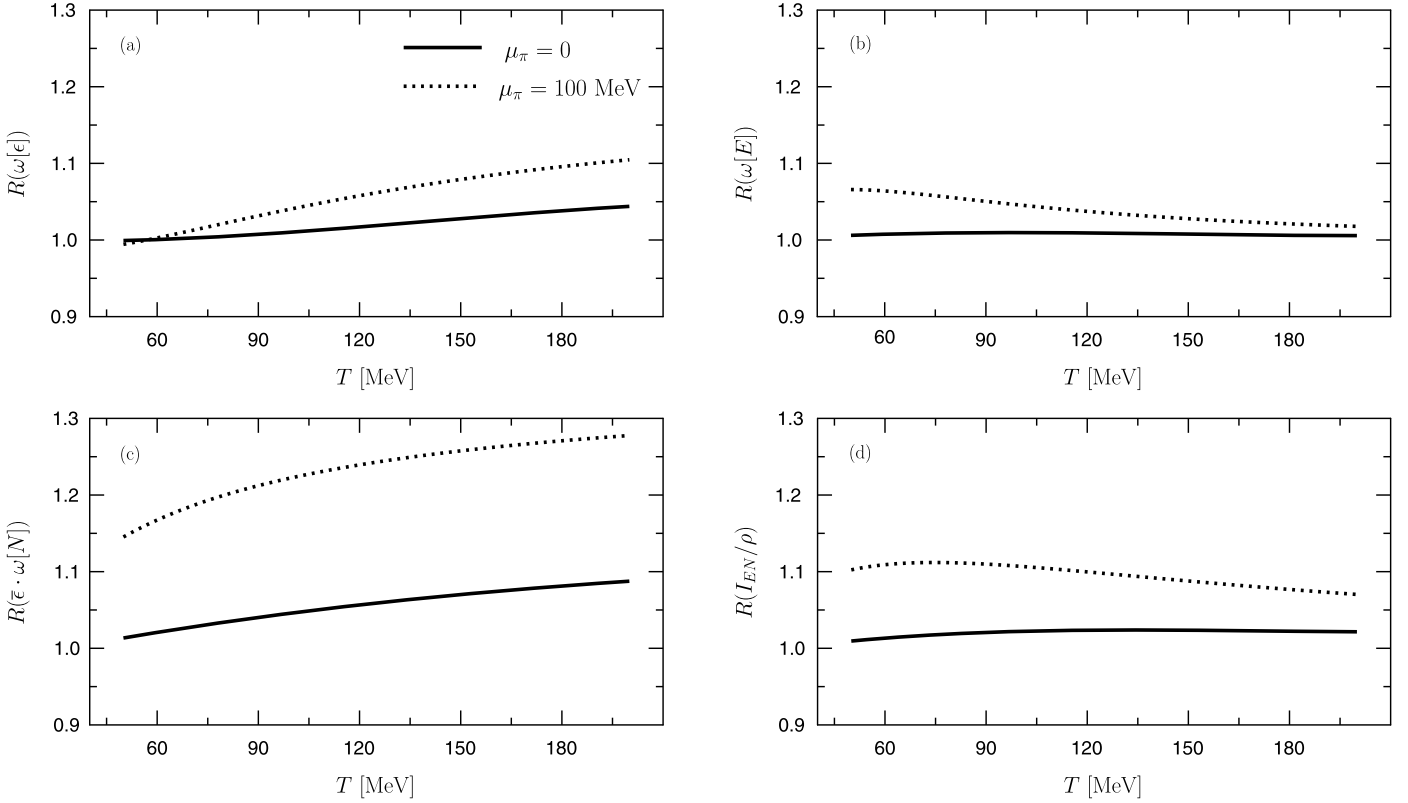


Fig. 2. The ratios (32) for quantities (a) $\omega[\epsilon]$, (b) $\omega[E]$, (c) $\bar{\epsilon} \cdot \omega[N]$, and (d) I_{EN}/ρ in the pion gas at $\mu_\pi = 0$ (solid lines) and $\mu_\pi = 100$ MeV (dotted lines).

3.3. Transverse momentum fluctuations

Using the above equations one can easily calculate the measures $\Delta[P_T, N]$ and $\Sigma[P_T, N]$ for the transverse momentum fluctuations. Since $p_T = p \cdot \sin(\theta)$ with $p = |\mathbf{p}|$ and θ being the angle between \mathbf{p} and the ‘beam’ z-axis, one gets for an arbitrary $f(p)$ function:

$$\begin{aligned} \int d^3p p_T f(p) &= \frac{\pi}{4} \int d^3p p f(p), \\ \int d^3p p_T^2 f(p) &= \frac{2}{3} \int d^3p p^2 f(p). \end{aligned} \quad (35)$$

First, one may calculate the statistical integrals in Eqs. (8)–(12) for P and N quantities, i.e. with p and p^2 instead of $\sqrt{p^2 + m^2}$ and $p^2 + m^2$, respectively. The expressions for ρ and for I_N remain unchanged. To calculate \bar{p}^2 and I_P the following relations can be used:

$$\bar{p}^2 = \bar{\epsilon}^2 - m^2, \quad I_P = I_E - m^2 \cdot I_N. \quad (36)$$

And there are two new integrals:

$$\bar{p} = \frac{d}{2\pi^2 \rho} \int_0^\infty p^2 dp \frac{p}{\exp[\sqrt{p^2 + m^2}/T] - 1}, \quad (37)$$

$$I_{PN} = \frac{d}{2\pi^2} \int_0^\infty p^2 dp \frac{p \exp[\sqrt{p^2 + m^2}/T]}{[\exp(\sqrt{p^2 + m^2}/T) - 1]^2}, \quad (38)$$

instead of $\bar{\epsilon}$ and I_{EN} , respectively.

Using Eq. (35) one then finds:

$$\bar{p}_T = \frac{\pi}{4} \cdot \bar{p}, \quad \bar{p}_T^2 = \frac{2}{3} \cdot \bar{p}^2, \quad \omega[p_T] = \frac{\bar{p}_T^2 - \bar{p}_T^2}{\bar{p}_T}, \quad (39)$$

$$I_{PT} = \frac{2}{3} \cdot I_E, \quad I_{PTN} = \frac{\pi}{4} \cdot I_{EN}. \quad (40)$$

The results for $\Delta[P_T, N]$ and $\Sigma[P_T, N]$ in the pion gas are shown in Fig. 3. From a comparison of this figure with Fig. 1 one concludes that qualitative behavior of Δ and Σ measures for extensive quantities $[P_T, N]$ is the same as for $[E, N]$ ones. Quantitatively, the Bose effects in Fig. 3 are smaller than the corresponding effects seen in Fig. 1. At $m = \mu = 0$ one obtains

$$\Delta[P_T, N] \cong \begin{pmatrix} 1.125 \\ 0.433 \end{pmatrix}, \quad \Sigma[P_T, N] \cong \begin{pmatrix} 0.931 \\ 1.398 \end{pmatrix}. \quad (41)$$

These values can be compared with the corresponding results for E and N , Eq. (30).

3.4. Connection to the Φ measure

The well-known fluctuation measure Φ was introduced in Ref. [8]. In a general case, when $X = \sum_{i=1}^N x_i$ represents any motional extensive quantity as a sum of single particle quantities, one gets [4]:

$$\Phi_X = \left[\frac{\omega[X] \langle X \rangle}{\langle N \rangle} \Sigma[X, N] \right]^{1/2} - [\bar{x}^2 - \bar{x}^2]^{1/2}, \quad (42)$$

where $\Sigma[X, N]$ is given by Eq. (3) and C_Σ by Eq. (4). Therefore, the Φ quantity can be expressed via measure Σ . At $m = \mu = 0$ it then follows:

$$\Phi_E = [\bar{\epsilon} \cdot \omega[\epsilon]]^{1/2} [(\Sigma[E, N])^{1/2} - 1] = \begin{pmatrix} -0.074 \\ 0.392 \end{pmatrix} T, \quad (43)$$

$$\Phi_{PT} = [\bar{p}_T \cdot \omega[p_T]]^{1/2} [(\Sigma[P_T, N])^{1/2} - 1] = \begin{pmatrix} -0.056 \\ 0.283 \end{pmatrix} T. \quad (44)$$

These results are in agreement with those obtained in Ref. [9].

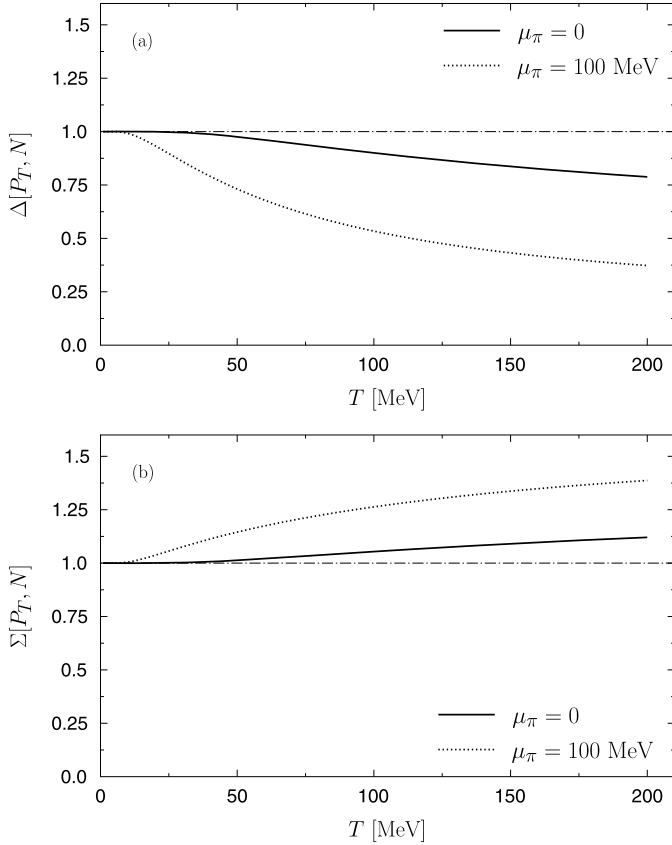


Fig. 3. The (a) $\Delta[P_T, N]$ and (b) $\Sigma[P_T, N]$ for the pion gas as the functions of T . The solid lines correspond to $\mu_\pi = 0$ and dotted lines to $\mu_\pi = 100$ MeV. The horizontal dashed lines show the Boltzmann approximation (19) equal to 1.

4. Summary

The strongly intensive fluctuation measures Δ and Σ have been studied for the ideal Bose and Fermi gases within the grand canonical ensemble. In the present Letter, the Δ and Σ quantities are considered for two specific extensive quantities – motional variable X (either the system energy E or transverse momentum P_T) and number of particles N . We have used the normalization of the strongly intensive measures which makes them dimensionless and equal to unity for fluctuations given by the independent particle model. The grand canonical ensemble within the Boltzmann approximation satisfies the conditions of independent particle model. Our results demonstrate deviations from the independent particle model due to the Bose and Fermi statistics. We present estimates of these quantum statistics effects for the hadron gas with thermodynamical parameters typical for the thermal models of A+A collisions. In the case of massless particles and zero chemical potential the Δ and Σ measures are calculated analytically. Numerical estimates for the Bose effects in the pion gas at the temperatures from 0 to 200 MeV are presented. For the Fermi gas of protons the quantum effects appear to be quite small.

The measures Δ and Σ are used to study the event-by-event fluctuations and correlations in high energy nucleus–nucleus and proton–proton collisions. From our results it follows that the Bose effects in the pion gas can be an important source of the transverse momentum fluctuations, especially in chemically non-equilibrium case with $\mu_\pi > 0$. However, other sources of dynamical fluctuations and correlations (e.g., exact conservation laws within micro-canonical ensemble, resonance decays, transverse collective flow, fluctuations of temperature, correlations between temperature and particle multiplicity, etc.) should be considered to make a realistic comparison of theoretical models with the data.

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